On supersolubility of finite groups admitting a Frobenius group of automorphisms with fixed-point-free kernel*

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Abstract

Assume that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. In this paper, we investigate this situation and prove that if $C_G(H)$ is supersoluble and $C_{G'}(H)$ is nilpotent, then G is supersoluble. Also, we show that G is a Sylow tower group of a certain type if $C_G(H)$ is a Sylow tower group of the same type.

1 Introduction

Throughout this paper, all groups mentioned are assumed to be finite. G always denotes a group, p denotes a prime, π denotes a set of primes, and $\mathbb{P}$ denotes the set of all primes. For any group G, we use the symbol $\pi(G)$ to denote the set of prime divisors of $|G|$.

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Recall that a Frobenius group FH with kernel F and complement H can be characterized as a group which is a semidirect product of a normal subgroup F by H such that $C_F(h) = 1$ for every non-identity element h of H. Recently, much research was focused on the case when a Frobenius group FH acts on a group G such that F acts fixed-point-freely, that is, $C_G(F) = 1$. It was shown that various properties of G are close to the corresponding properties of $C_G(H)$ in this situation, see [4–7, 9–12]. For instance, E. I. Khukhro, N. Y. Makarenko and P. Shumyatsky [9] proved that the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$, and G is nilpotent if $C_G(H)$ is nilpotent; E. I. Khukhro [7] established that the Fitting height of G is equal to the Fitting height of $C_G(H)$; P. Shumyatsky [11] proved that if F is cyclic and $C_G(H)$ satisfies a positive law of degree k, then G satisfies a positive law of degree that is bounded solely in terms of k and $|FH|$.

The main aim of this paper is to discuss the problem which was proposed by E. I. Khukhro in Fourth Group Theory Conference of Iran (see [8]). In the above situation, he asked that if $C_G(H)$ is supersoluble, whether a group G is supersoluble or not. Though this problem has not been solved, we can give a positive answer if we suppose further that $C_{G'}(H)$ is nilpotent. In fact, a more generalized result is obtained. In Section 3, we prove that G is p-supersoluble if $C_G(H)$ is p-supersoluble and $C_{G'}(H)$ is p-nilpotent. Moreover, we show that G is a Sylow tower group of a certain type if $C_G(H)$ is a Sylow tower group of the same type.

2 Preliminaries

The following results are useful in our proof.

Lemma 2.1. *(see [2, Theorem 0.11].) Suppose that a group G admits a nilpotent group of automorphisms F such that $C_G(F) = 1$. Then G is soluble.*

Lemma 2.2. *(see [9, Lemma 2.2].) Let G be a group admitting a nilpotent group of automorphisms F such that $C_G(F) = 1$. If N is an F-invariant normal subgroup of G, then $C_{G/N}(F) = 1$.*

Lemma 2.3. *(see [9, Lemma 2.3].) Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H. If N is an FH-invariant normal subgroup of G such that $C_N(F)$=1, then $C_{G/N}(H)$= $C_G(H)N/N$.*

Lemma 2.4. *(see [7, Lemma 2.2].) Let FH be a Frobenius group with kernel F and complement H. In any action of FH with nontrivial action of F, the complement H acts faithfully.*

Lemma 2.5. *Let G be a non-trivial group admitting a Frobenius group of actions FH with kernel F and complement H and K be the kernel of FH acts on G. If $C_G(F) = 1$, then $K < F$ and FH/K is a Frobenius group with kernel F/K and complement HK/K.*

Proof. As $F \nleq K$, we have that $H \cap K = 1$ by Lemma 2.4. Hence $K \leq F$ because $(|F|, |H|) = 1$. Then for every non-trivial element $h \in H$, since $(|F|, |\langle h \rangle|) = 1$, $C_{F/K}(h) = C_F(h)K/K = 1$. This implies that FH/K is a Frobenius group with kernel F/K and complement HK/K. $\square$

Recall that for a soluble group G, the Fitting series starts with $F_0(G) = 1$, followed by the Fitting subgroup $F_1(G) = F(G)$, and $F_{i+1}(G)$ is defined as the inverse image of $F(G/F_i(G))$. The next lemma is a collection of [7, Theorem 2.1 and Corollary 4.1] and [9, Lemma 2.4 and Theorem 2.7].

Lemma 2.6. *Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then:*

(1) $|G| = |C_G(H)|^{|H|}$.

(2) $G = \langle C_G(H)^f \mid f \in F \rangle$.

(3) *If $C_G(H)$ is nilpotent, then G is nilpotent.*

(4) $F_i(C_G(H)) = F_i(G) \cap C_G(H)$.

(5) $O_\pi(C_G(H)) = O_\pi(G) \cap C_G(H)$ *for any set of primes π.*

In the following lemma, the symbols $\mathfrak{U}$ and $\mathfrak{A}(p-1)$ denote the class of all supersoluble groups and the class of all abelian groups of exponent dividing $p-1$, respectively. Also, a normal subgroup N of G is called $\mathfrak{U}$-hypercentral in G if either $N = 1$ or $N > 1$ and all chief factors of G below N are cyclic. Let $Z_\mathfrak{U}(G)$ denote the $\mathfrak{U}$-hypercentre of G, that is, the product of all $\mathfrak{U}$-hypercentral normal subgroups of G.

Lemma 2.7. *Let E be a normal p-subgroup of a group G. Then $E \leq Z_\mathfrak{U}(G)$ if and only if $(G/C_G(E))^{\mathfrak{A}(p-1)} \leq O_p(G/C_G(E))$.*

Proof. The necessity directly follows from [13, Lemma 2.2]. Now we prove the sufficiency. Since $O_p(G/C_G(H/K)) = 1$ for any chief factor H/K of G below E and $C_G(E) \leq C_G(H/K)$, $G/C_G(H/K) \leq \mathfrak{A}(p-1)$. Hence $|H/K| = p$ by [13, Lemma 2.1]. This shows that $E \leq Z_\mathfrak{U}(G)$. $\square$

3 Main Results

Firstly, we begin to show the connection between the properties of G and $C_G(H)$ by proving that G is p-closed (resp. p-nilpotent) if $C_G(H)$ is p-closed (resp. p-nilpotent).

Theorem 3.1. *Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. If $C_G(H)$ is p-closed, then G is p-closed.*

Proof. Suppose that the theorem is not true. Then we may consider a counterexample G of minimal order. We proceed the proof via the following steps:

(1) *$F(G) = O_q(G)$, where q is a prime such that $q \neq p$, and $G/O_q(G)$ is p-closed.*

By Lemma 2.1, G is soluble, and so $F(G) \neq 1$. Let q be any prime dividing $|F(G)|$. Then $G/O_q(G)$ is FH-invariant. In view of Lemmas 2.2, 2.3 and 2.5, it is easy to see that the hypothesis of the theorem still holds for $G/O_q(G)$. By the minimality of our counterexample, we have that $G/O_q(G)$ is p-closed. If $q = p$, then G is p-closed, a contradiction. Thus $q \neq p$. Now assume that there exists another prime r dividing $|F(G)|$. Then with the same argument as above, $G/O_r(G)$ is p-closed, and consequently, G is p-closed. This contradiction shows that $F(G) = O_q(G)$.

(2) *$G = O_q(G)P$, where P is a Sylow p-subgroup of G.*

By (1), $O_q(G)P$ is an FH-invariant normal subgroup of G. In view of Lemma 2.5, $O_q(G)P$ satisfies the hypothesis of the theorem. If $O_q(G)P < G$, then by the minimality of our counterexample, we have that $O_q(G)P$ is p-closed, and so G is p-closed, a contradiction. Hence $G = O_q(G)P$.

(3) *The final contradiction.*

Obviously, G is q-closed by (2), and so $C_G(H)$ is nilpotent. Then by Lemma 2.6(3), G is nilpotent. The final contradiction finishes the proof. □

Theorem 3.2. *Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. If $C_G(H)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Suppose that the theorem does not hold. Let G be a counterexample of minimal order. Then:

(1) *$O_{p'}(G) = 1$, and so $F(G) = O_p(G)$.*

By Lemmas 2.2, 2.3 and 2.5, $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. If $O_{p'}(G) \neq 1$, then by our choice, $G/O_{p'}(G)$ is p-nilpotent, and thereby G is p-nilpotent, a contradiction. Thus $O_{p'}(G) = 1$, and so $F(G) = O_p(G)$.

(2) *$G = O_p(G)Q$, where Q is the unique FH-invariant Sylow q-subgroup of G with $q \neq p$.*

By [9, Lemma 2.6], there exists a unique FH-invariant Sylow q-subgroup of G, denoted by Q. If $O_p(G)Q < G$, then $O_p(G)Q$ satisfies the hypothesis of the theorem by Lemma 2.5, and so $O_p(G)Q$ is p-nilpotent by the minimality of our counterexample. This yields that $O_p(G)Q$

is nilpotent. Then $Q \le C_G(O_p(G))$. Since G is soluble by Lemma 2.1, $C_G(F(G)) \le F(G)$. This implies that $Q \le F(G)$, which contradicts (1). Hence $G = O_p(G)Q$.

(3) *The final contradiction.*

Since $C_G(H)$ is p-nilpotent and G is p-closed by (2), we have that $C_G(H)$ is nilpotent, which forces that G is also nilpotent by Lemma 2.6(3). This is the final contradiction. $\square$

The following lemma can be viewed as not only an improvement of [9, Lemma 2.6], but also a key step in the proof of Theorem 3.4.

Lemma 3.3. *Suppose that a group G admits a Frobenius group of automorphisms with kernel F and complement H such that $C_G(F) = 1$. Then for any subset of primes π of $\pi(G)$, there exists a unique FH-invariant Hall π-subgroup of G. Furthermore, the set of all unique FH-invariant Hall subgroups forms a Hall system of G.*

Proof. By Lemma 2.1, G is soluble, and so is GF. Since $C_G(F) = 1$, it is easy to see that F is a Carter subgroup of GF. Then F contains a system normalizer of GF by [3, Chapter V, Theorem 4.1]. By [3, Chapter I, Theorem 5.6], a system normalizer covers all central chief factors of GF, and so F is a system normalizer of GF because F is nilpotent. This implies that there exists an F-invariant Hall π-subgroup S of G. If S and S^g are both F-invariant Hall π-subgroups of G, where $g \in G$, then F and $F^{g^{-1}}$ are two Carter subgroups of $N_{GF}(S) = N_G(S)F$. Hence $F = F^{g^{-1}g'}$ for some $g' \in N_G(S)$. Since $N_G(F) = C_G(F) = 1$, $g^{-1}g' = 1$, and so $g \in N_G(S)$. Thus $S = S^g$. This shows that S is the unique F-invariant Hall π-subgroup of G. Since S^h is F-invariant for any $h \in H$, we have that S is also H-invariant. Furthermore, let $\pi(G) = \{p_1, \ldots, p_r\}$ and S_i be the unique FH-invariant Sylow p_i-complement of G for $1 \le i \le r$. Note that for every subset of primes π of $\pi(G)$, $G_\pi = \bigcap\{S_i : p_i \in \pi(G)\backslash\pi\}$ is the unique FH-invariant Hall π-subgroup of G. Then by [3, Chapter I, Proposition 4.4], the set of all unique FH-invariant Hall subgroups forms a Hall system of G. $\square$

Now we can establish our main result as follows.

Theorem 3.4. *Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. If $C_G(H)$ is p-supersoluble and $C_{G'}(H)$ is p-nilpotent, then G is p-supersoluble.*

Proof. Suppose that the theorem is not true. Then we may consider a counterexample G of minimal order. Then:

(1) *Let S be an FH-invariant proper subgroup of G and N be a non-trivial FH-invariant normal subgroup of G. Then S and G/N are both p-supersoluble.*

By the minimality of our counterexample, it is sufficient to prove that S and G/N both satisfy the hypothesis of the theorem. Clearly, S satisfies the hypothesis of the theorem by Lemma 2.5. Let $C_{G'N/N}(H) = A/N$ and $C_{G'/G'\cap N}(H) = B/G'\cap N$. Note that by Lemma 2.3, $A/N = BN/N = C_{G'}(H)N/N$ is p-nilpotent and $C_{G/N}(H) = C_G(H)N/N$ is p-supersoluble. Hence by Lemmas 2.2 and 2.5, G/N also satisfies the hypothesis of the theorem.

(2) $O_{p'}(G) = O_{p'}(C_G(H)) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Then by (1), $G/O_{p'}(G)$ is p-supersoluble. Thus G is p-supersoluble. This contradiction shows that $O_{p'}(G) = 1$, which forces that $O_{p'}(C_G(H)) = 1$ by Lemma 2.6(5).

(3) *$F_p(G) = F(G) = P$ is the socle of G, where $F_p(G)$ denotes the largest normal p-nilpotent subgroup of G and P denotes the normal Sylow p-subgroup of G, and $C_G(P) = P$.*

Since $C_G(H)$ is p-supersoluble and $O_{p'}(C_G(H)) = 1$ by (2), $C_G(H)$ is p-closed by [1, Lemma 2.1.6]. It follows from Theorem 3.1 that G is p-closed. As $O_{p'}(G) = 1$ by (2), $F_p(G) = F(G) = P$, where P is the normal Sylow p-subgroup of G. If $\Phi(G) \neq 1$, then $G/\Phi(G)$ is p-supersoluble by (1). This implies that G is p-supersoluble, which is impossible. Thus $\Phi(G) = 1$, and so P is the socle of G. Note that by Lemma 2.1, G is soluble. It follows that $C_G(P) \leq P$. Therefore, we have that $C_G(P) = P$.

(4) *G has the unique FH-invariant Hall p'-subgroup T and T is abelian.*

By Lemma 2.5, G' satisfies the hypothesis of Theorem 3.2. Thus G' is p-nilpotent, and so $G' \leq P$ by (3). This implies that G/P is abelian. In view of Lemma 3.3, G has the unique FH-invariant Hall p'-subgroup T, and clearly, T is abelian.

(5) *The exponent of $C_T(H)$ divides $p-1$.*

Let $C = P \cap C_G(H)$. Then obviously, C is the normal Sylow p-subgroup of $C_G(H)$. By (3) and Lemma 2.6(4), $F(C_G(H)) = C$, and so $C_{C_G(H)}(C) = C$. Since $C_G(H)$ is p-supersoluble, $C \leq Z_{\mathfrak{U}}(C_G(H))$, and consequently, $C_G(H)/C = C_G(H)/C_{C_G(H)}(C) \in \mathfrak{A}(p-1)$ by Lemma 2.7. This implies that the exponent of $C_T(H)$ divides $p-1$.

(6) *The final contradiction.*

By applying Lemmas 2.5 and 2.6(2) for T, we have that $T = \langle C_T(H)^f | f \in F\rangle$. Since T is abelian by (4), the exponent of $C_T(H)$ is equal to the exponent of T. Then by (5), the exponent of T divides $p-1$, and so $T \in \mathfrak{A}(p-1)$ by (4). Hence by (3), $G/C_G(P) = G/P \cong T \in \mathfrak{A}(p-1)$, which yields that $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.7. Thus G is p-supersoluble. The final contradiction finishes the proof. □

From Theorem 3.4, we can directly deduce the next corollary.

Corollary 3.5. *Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. If $C_{G'}(H)$ is nilpotent and $C_G(H)$ is supersoluble, then G is supersoluble.*

Recall that if σ denotes a linear ordering on $\mathbb{P}$, then a group G is called a Sylow tower group of type σ if there exists a series of normal subgroups of G: $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that G_i/G_{i-1} is a Sylow p_i-subgroup of G/G_{i-1} for $1 \leq i \leq n$, where $p_1 \prec p_2 \prec \cdots \prec p_n$ is the ordering induced by σ on the distinct prime divisors of $|G|$. Here we arrive at the next theorem.

Theorem 3.6. *Suppose that a group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. If $C_G(H)$ is a Sylow tower group of a certain type, then G is a Sylow tower group of the same type.*

Proof. Suppose that $C_G(H)$ is a Sylow tower group of type σ and $p_1 \prec p_2 \prec \cdots \prec p_r$ is the ordering induced by σ on the distinct prime divisors of $|G|$. Then by Lemma 2.6(1), p_i divides $|C_G(H)|$ for $1 \leq i \leq r$. Since $C_G(H)$ is a Sylow tower group of type σ, $C_G(H)$ is p_1-closed, and so G is p_1-closed by Theorem 3.1. Let G_1 be the normal Sylow p_1-subgroup of G. Then clearly, G/G_1 is FH-invariant. Since G/G_1 satisfies the hypothesis of the theorem by Lemmas 2.2, 2.3 and 2.5, by induction, G/G_1 is a Sylow tower group of type σ, and so is G. □

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